The Algebraic footprints of another Tensor Monster: The Combinatorial Delta (By R. J. Cano, Jan 9 2013)

Consider that you have two unidimensional arrays A, B and you need to multiply each element in such lists: For example each element in A for each one of the elements in B, and then after all those products are formed, you sum them. By doing this you might now distinguish between two kind of quadratic terms where: 1) All the factors have the same indices. 2) All the factors have distinct indices.

You might also assume that both vectors, unidimensional arrays or lists are of the same size meaning this the quantity of elements composing them. If such condition were not already provided, you can fix it up taking and padding with zeros the smaller list.

Let be m the common size for both lists[1], there are always exactly m terms of the (1) kind described before, this is, terms where both factors have the same index, and also exactly $(\eta = m - 1)$:

$$\begin{aligned}
\left(m^2 - m\right) &= (m-1)\,m \\
&= \eta\,[\eta + 1] \\
&= 2\left(\frac{1}{2}\eta\,[\eta+1]\right) \\
&= 2\sum_{k=1}^{\eta} k \\
&= 2\,[1 + 2 + 3 + 4 + \cdots + (m-2) + (m-1)]
\end{aligned} \tag{1}$$

As many elements of the (2) kind previously described as the double of non zero entries that has a strictly upper square matrix of size m.

And we have invoked here at least a pair of key concepts. If you look carefully at the non zero entries of a strictly upper square matrix regardless its size m, you will notice that the indices of such entries form the $C\,(m, 2)$ possible binary combinations for the first m natural numbers. Since $2 = 2!$, there is no error by interpreting (at least for this moment) the eq. 1 (above) also as:

$$\begin{aligned}
m^2 - m &= 2C\,(m, 2) \\
&= 2!C\,(m, 2) = P\,(m, 2)
\end{aligned} \tag{2}$$

[1] After being fixed up if needed.

Therefore, by substitution we have:

$$\begin{aligned} P\,(m,2) &= 2\,[1+2+3+4+\cdots+(m-2)+(m-1)] \\ C\,(m,2) &= [1+2+3+4+\cdots+(m-2)+(m-1)] \end{aligned} \tag{3}$$

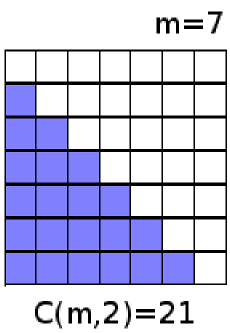
Both identities in eq. 3 give place for the following: Traditionally, the so called binary combinations are defined in terms of products as:

$$C\,(m,2) = \frac{P\,(m,2)}{2} = \frac{m(m-1)}{2} \tag{4}$$

Then, this is a good question: There might exist a general kind of summation for getting the count of the Q-ary possible combinations for the first m natural numbers $(Q \leq m)$???

The answer of such question lies on all the modern buildings where one walks daily: The stairways.

Let's take a look again on a strictly upper square matrix of size m. Its transpose is strictly lower and if it were represented graphically as a square divided in m^2 identical parts where the corresponding non zero entries are painted in blue, then we would see something like this:



Where the identity $m^2 = m + 2 \times C(m,2)$ becomes obvious. Also in this picture from left to right it might be noticed that the total count of blue blocks can be computed alternatively (there $\mathbb{U}$ is the unit matrix or 1 everywhere) by:

$$C\,(m,2) = \sum_{i=1}^{(m-1)} \sum_{j=(i+1)}^{m} \mathbb{U}_{ij} \tag{5}$$

And it works because the expansion of the right side in eq. (5) adds a unit to the final result for each possible binary combination of the indices i and j.

More interesting yet. Now by substitution between equations (4) and (5) we have:

$$\frac{(m-1)\,m}{2!} = \sum_{i=1}^{(m-1)} \sum_{j=(i+1)}^{m} \mathbb{U}_{ij} \tag{6}$$

Identified the second member of an infinite family of identities. The first would be:

$$\frac{m}{1!} = \sum_{k=1}^{m} \mathbb{U}_{kk} = Tr\left(\tilde{I}\right)$$

Where $\tilde{I}$ is the identity matrix of size m. The third member is:

$$\frac{(m-2)\,(m-1)\,m}{3!} = \sum_{i=1}^{(m-2)} \sum_{j=(i+1)}^{(m-1)} \sum_{k=(j+1)}^{m} \mathbb{U}_{ijk} \tag{7}$$

And in general the r-th member might be written as:

$$\frac{\prod_{s=1}^{r}(m-s+1)}{r!} = \sum_{J(1)=1}^{m-r+1} \sum_{J(2)=J(1)+1}^{m-r+2} \sum_{J(3)=J(2)+1}^{m-r+3} \cdots \sum_{J(r-1)=J(r-2)+1}^{m-\cancel{r+}(r-1)} \sum_{J(r)=J(r-1)+1}^{m-\cancel{r+r}} \tag{8}$$

Where it was omitted for simplicity the $\mathbb{U}_{\{J(k)\}}$ with k in $[1..r]$ since it is defined always as the unit everywhere.

Why these identities are important?: Although in each one of these identities the difference between the left side and the right side would be the same difference between using either the stairways or an elevator, these identities confirm the elementary connection between the algebra of indices and the combinatorics. Also this family enables us to build identities similar to the multinomial theorem but more flexibles, like this:[2]

$$\left[\sum_{i=1}^{m} A_i\right] \cdot \left[\sum_{j=1}^{m} B_j\right] = \left[\sum_{k=1}^{m} A_k B_k\right] + 2 \cdot \left[\sum_{v=1}^{(m-1)} \sum_{w=(v+1)}^{m} A_v B_w\right] \tag{9}$$

[2]An extremely useful identity for the Mathematical methods in Physics. Even when B_j be a partial derivative.